

OPTIMAL BLOW UP RATE FOR THE CONSTANTS OF KHINCHIN TYPE INEQUALITIES

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ABSTRACT. We provide, among other results, the optimal blow up rate of the constants of a family of Khinchin inequalities for multiple sums.

1. INTRODUCTION

The Khinchin inequality was designed in 1923 by A. Khinchin ([8]) to estimate the asymptotic behavior of certain random walks. The following example provides an illustration of its reach. Suppose that you have n real numbers a_1, \dots, a_n and a fair coin. When you flip the coin, if it comes up heads, you chose $\alpha_1 = a_1$, and if it comes up tails, you choose $\alpha_1 = -a_1$. After having flipped the coin k times you have the number

$$\alpha_{k+1} := \alpha_k + a_{k+1},$$

if it comes up heads and

$$\alpha_{k+1} := \alpha_k - a_{k+1},$$

if it comes up tails. After completed all n steps, what should be the expected value of

$$|\alpha_n| = \left| \sum_{k=1}^n \pm a_k \right|?$$

Khinchin's inequality, in some sense, solves this question. Nowadays it is a very important probabilistic tool with deep inroads in Mathematical Analysis and Banach Space Theory. It asserts that for any $p > 0$ there are constants $A_p, B_p > 0$ such that

$$(1.1) \quad A_p \left(\sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}} \leq \left(\int_0^1 \left| \sum_{j=1}^n r_j(t) a_j \right|^p dt \right)^{\frac{1}{p}} \leq B_p \left(\sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}}$$

for all sequence of scalars $(a_i)_{i=1}^n$ and all positive integers n . Above, as usual, $(r_n : [0, 1] \rightarrow \mathbb{R})_{n=1}^\infty$ is a sequence of independent and identically distributed random variables defined by

$$r_n(t) := \text{sign}(\sin 2^n \pi t),$$

called Rademacher functions. It is folklore that the optimal constants A_p, B_p are the same for real and complex scalars, so it suffices to work with real scalars. It was proved by Szarek ([14]) that $A_1 = (\sqrt{2})^{-1}$ is optimal, solving a long standing problem posed by Littlewood (see [7]). Later, Haagerup ([6]) simplified Szarek's approach and provided the optimal constants for $p \neq 1$ (see also [9, 15, 16]).

The Khinchin inequality is also valid – and useful – for multiple sums. It is well-known (see [13]) that regardless of the choice of the positive integers m, n and scalars $a_{i_1, \dots, i_m}, i_1, \dots, i_m =$

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$1, \dots, n$, we have

$$(1.2) \quad \left(\sum_{i_1, \dots, i_m=1}^n |a_{i_1, \dots, i_m}|^2 \right)^{\frac{1}{2}} \leq A_p^{-m} \left(\int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^n a_{i_1, \dots, i_m} r_{i_1}(t_1) \cdots r_{i_m}(t_m) \right|^p dt_1 \cdots dt_m \right)^{\frac{1}{p}}.$$

We stress that even in the simple case $m = 2$, the sequence of random variables $(r_{i_1} \cdot r_{i_2} : [0, 1]^2 \rightarrow \mathbb{R})_{n, m=1}^\infty$ is not independent.

In the present paper, among other results, we provide the exact blow up rate of the constants in (1.2) as n grows when the ℓ_2 -norm in the left-hand-side is replaced by an ℓ_r -norm with $0 < r < 2$. More precisely, we prove the following:

Theorem 1. *Let m, n be positive integers and $(a_{i_1, \dots, i_m})_{i_1, \dots, i_m=1}^n$ be a sequence of real scalars. If $0 < r < 2$, then there is a constant $C_{m,p} > 0$ such that*

$$\left(\sum_{i_1, \dots, i_m=1}^n |a_{i_1, \dots, i_m}|^r \right)^{\frac{1}{r}} \leq C_{m,p} n^{m(\frac{1}{r} - \frac{1}{2})} \left(\int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^n a_{i_1, \dots, i_m} r_{i_1}(t_1) \cdots r_{i_m}(t_m) \right|^p dt_1 \cdots dt_m \right)^{\frac{1}{p}}$$

and the exponent $m(\frac{1}{r} - \frac{1}{2})$ is optimal.

The main technicality in the proof of the above result arises in the search of the optimality of the parameters. For this task we shall use, among other results, a powerful and deep combinatorial probabilistic tool, called Kahane–Salem–Zygmund inequality.

2. PRELIMINARIES

We start off by recalling some terminology. By c_0 we denote the Banach space of all real-valued sequences $(a_j)_{j=1}^\infty$ such that $\lim_{j \rightarrow \infty} a_j = 0$, endowed with the sup norm. For a multilinear form $T : c_0 \times \cdots \times c_0 \rightarrow \mathbb{R}$ we denote, as usual,

$$\|T\| := \sup \left\{ \left| T(x^{(1)}, \dots, x^{(m)}) \right| : \|x^{(j)}\| = 1 \text{ for all } j = 1, \dots, m \right\}.$$

For more details on the theory of multilinear forms on Banach spaces we refer to [10]. For the reader's convenience we also recall that the topological dual of c_0 , denoted by $(c_0)^*$ is isometrically isomorphic to the sequence space of absolutely summable sequences ℓ_1 .

We shall recall three important tools of Probability Theory and multilinear operators that will be crucial to prove Theorem 1 and Proposition 1. The first one is the beautiful Kahane–Salem–Zygmund inequality (see, for instance, [3] and [4] and the references therein):

Theorem 2 (Kahane–Salem–Zygmund inequality). *Let $m, n \geq 1$. There is a universal constant $K_m > 0$, depending only on m , and an m -linear form $T_{m,n} : c_0 \times \cdots \times c_0 \rightarrow \mathbb{R}$ of the form*

$$T_{m,n}(z^{(1)}, \dots, z^{(m)}) = \sum_{i_1, \dots, i_m=1}^n \pm z_{i_1}^{(1)} \cdots z_{i_m}^{(m)}$$

such that

$$\|T_{m,n}\| \leq K_m n^{\frac{m+1}{2}}.$$

As it will be seen in the next section, we shall prove the optimality of Theorem 1 by considering, for all i_{m+1} ,

$$a_{i_1 \dots i_m}^{(i_{m+1})} = T_{m+1,n}(e_{i_1}, \dots, e_{i_{m+1}}).$$

To the proof the optimality of Proposition 1 we shall need a different approach. We shall consider m -linear forms $R_m : c_0 \times \cdots \times c_0 \rightarrow \mathbb{R}$ defined inductively by

$$\begin{aligned} R_2(x^{(1)}, x^{(2)}) &= x_1^{(1)} x_1^{(2)} + x_1^{(1)} x_2^{(2)} + x_2^{(1)} x_1^{(2)} - x_2^{(1)} x_2^{(2)}, \\ R_3(x^{(1)}, x^{(2)}, x^{(3)}) &= \left(x_1^{(1)} + x_2^{(1)} \right) \left(x_1^{(2)} x_1^{(3)} + x_1^{(2)} x_2^{(3)} + x_2^{(2)} x_1^{(3)} - x_2^{(2)} x_2^{(3)} \right) \\ &\quad + \left(x_1^{(1)} - x_2^{(1)} \right) \left(x_3^{(2)} x_3^{(3)} + x_3^{(2)} x_4^{(3)} + x_4^{(2)} x_3^{(3)} - x_4^{(2)} x_4^{(3)} \right), \end{aligned}$$

and so on (for details we refer to [11]), and consider, for all i_{m+1} ,

$$a_{i_1 \dots i_m}^{(i_{m+1})} = R_{m+1}(e_{i_1}, \dots, e_{i_{m+1}}).$$

It shall be important to note (see [11]) that each R_m is composed by precisely 2^{2m-2} monomials and that

$$\|R_m\| = 2^{m-1}.$$

It is also important for our purposes to note that each R_m has exactly 2^{m-1} monomials involving the coordinates of the last variable $x^{(m)}$.

Finally, we need a “multiple index” version of the Contraction Principle. We present a proof for the sake of completeness.

Lemma 1. *For all positive integers m, n and vectors y_{i_1, \dots, i_m} in a Banach space Y , $i_1, \dots, i_m = 1, \dots, n$, we have*

$$\max_{\substack{i_k=1, \dots, n \\ k=1, \dots, m}} \|y_{i_1, \dots, i_m}\| \leq \int_{[0,1]^m} \left\| \sum_{i_1, \dots, i_m=1}^n r_{i_1}(t_1) \cdots r_{i_m}(t_m) y_{i_1, \dots, i_m} \right\| dt_1 \cdots dt_m.$$

Proof. The case $m = 1$ is the Contraction Principle (see [5, Theorem 12.2]). Let us suppose, as the induction step, that the result is valid for $m - 1$. Thus, for all positive integers i_1, \dots, i_m , we have

$$\begin{aligned} & \int_{[0,1]^m} \left\| \sum_{i_1, \dots, i_m=1}^n r_{i_1}(t_1) \cdots r_{i_m}(t_m) y_{i_1, \dots, i_m} \right\| dt_1 \cdots dt_m \\ &= \int_{[0,1]^{m-1}} \left(\int_0^1 \left\| \sum_{i_1=1}^n r_{i_1}(t_1) \left(\sum_{i_2, \dots, i_m=1}^n r_{i_2}(t_2) \cdots r_{i_m}(t_m) y_{i_1, \dots, i_m} \right) \right\| dt_1 \right) dt_2 \cdots dt_m \\ &\geq \int_{[0,1]^{m-1}} \left\| \sum_{i_2, \dots, i_m=1}^n r_{i_2}(t_2) \cdots r_{i_m}(t_m) y_{i_1, \dots, i_m} \right\| dt_2 \cdots dt_m \\ &\geq \|y_{i_1, \dots, i_m}\|. \end{aligned}$$

□

3. THE PROOF OF THE MAIN THEOREM

Let us first show that there is a $t_{m,p} > 0$ and a certain constant $C_{m,p} > 0$ such that

$$\begin{aligned} (3.1) \quad & \left(\sum_{i_1, \dots, i_m=1}^n |a_{i_1, \dots, i_m}|^r \right)^{\frac{1}{r}} \\ & \leq C_{m,p} n^{t_{m,p}} \left(\int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^n r_{i_1}(t_1) \cdots r_{i_m}(t_m) a_{i_1, \dots, i_m} \right|^p dt_1 \cdots dt_m \right)^{1/p} \end{aligned}$$

for all sequences $(a_{i_1, \dots, i_m})_{i_1, \dots, i_m=1}^n$ and all n .

Let $s > 0$ be such that $\frac{1}{r} = \frac{1}{2} + \frac{1}{s}$. By the Hölder inequality and (1.2) with $p = 1$ we have

$$\begin{aligned} \left(\sum_{i_1, \dots, i_m=1}^n |a_{i_1, \dots, i_m}|^r \right)^{\frac{1}{r}} &\leq \left(\sum_{i_1, \dots, i_m=1}^n |a_{i_1, \dots, i_m}|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i_1, \dots, i_m=1}^n 1^s \right)^{\frac{1}{s}} \\ &\leq 2^{\frac{m}{2}} \cdot \left(\int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^n r_{i_1}(t_1) \cdots r_{i_m}(t_m) a_{i_1, \dots, i_m} \right| dt_1 \cdots dt_m \right) \cdot n^{\frac{m}{s}} \\ &= 2^{\frac{m}{2}} n^{m(\frac{1}{r} - \frac{1}{2})} \cdot \int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^n r_{i_1}(t_1) \cdots r_{i_m}(t_m) a_{i_1, \dots, i_m} \right| dt_1 \cdots dt_m. \end{aligned}$$

Now we show that the best estimate for $t_{m,1}$ in (3.1) is precisely $m(\frac{1}{r} - \frac{1}{2})$. In fact, let $T_{m+1,n}$ be given by the Kahane–Salem–Zygmund inequality (Theorem 2). Since $(c_0)^* = \ell_1$ we have

$$\begin{aligned} &\sum_{i_{m+1}=1}^n \left(\sum_{i_1, \dots, i_m=1}^n |T_{m+1,n}(e_{i_1}, \dots, e_{i_{m+1}})|^r \right)^{\frac{1}{r}} \\ &\leq \sum_{i_{m+1}=1}^n C_{m,p} n^{t_{m,1}} \int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^n r_{i_1}(t_1) \cdots r_{i_m}(t_m) T_{m+1,n}(e_{i_1}, e_{i_2}, \dots, e_{i_{m+1}}) \right| dt_1 \cdots dt_m \\ &= C_{m,p} n^{t_{m,1}} \int_{[0,1]^m} \sum_{i_{m+1}=1}^n \left| T_{m+1,n} \left(\sum_{i_1=1}^n r_{i_1}(t_1) e_{i_1}, \dots, \sum_{i_m=1}^n r_{i_m}(t_m) e_{i_m}, e_{i_{m+1}} \right) \right| dt_1 \cdots dt_m \\ &\leq C_{m,p} n^{t_{m,1}} \sup_{t_1, \dots, t_m \in [0,1]} \left\| T_{m+1,n} \left(\sum_{i_1=1}^n r_{i_1}(t_1) e_{i_1}, \dots, \sum_{i_m=1}^n r_{i_m}(t_m) e_{i_m}, \cdot \right) \right\| \\ &\leq C_m n^{t_{m,1}} K_{m+1} n^{\frac{m+2}{2}}. \end{aligned}$$

On the other hand,

$$\sum_{i_{m+1}=1}^n \left(\sum_{i_1, \dots, i_m=1}^n |T_{m+1,n}(e_{i_1}, \dots, e_{i_{m+1}})|^r \right)^{\frac{1}{r}} = n \cdot n^{\frac{m}{r}}.$$

Hence

$$n^{1+\frac{m}{r}} \leq C_{m,p} n^{t_{m,1}} K_{m+1} n^{\frac{m+2}{2}}$$

for all n . Since n is arbitrary, we have

$$t_{m,1} \geq m \left(\frac{1}{r} - \frac{1}{2} \right).$$

By [13] we know that for any $p, q > 0$ and all positive integers m , there is a constant $C_{m,p,q} > 0$ such that

$$\begin{aligned} &\left(\int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^n a_{i_1, \dots, i_m} \prod_{j=1}^m r_{i_j}(t_j) \right|^p dt_1 \cdots dt_m \right)^{\frac{1}{p}} \\ &\leq C_{m,p,q} \left(\int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^n a_{i_1, \dots, i_m} \prod_{j=1}^m r_{i_j}(t_j) \right|^q dt_1 \cdots dt_m \right)^{\frac{1}{q}} \end{aligned}$$

and thus we conclude that the optimal $t_{m,p}$ coincides with the optimal $t_{m,1}$, regardless of the $p > 0$, and the proof is done.

Remark 1. If $0 < r_j < 2$ for all $j = 1, \dots, m$, using the mixed Hölder inequality (see [2]) and repeating the arguments of the proof of Theorem 1 we can prove that there is a constant $C_{m,p} > 0$ such that

$$\begin{aligned} & \left(\sum_{i_1=1}^n \left(\sum_{i_2=1}^n \left(\cdots \left(\sum_{i_m=1}^n |a_{i_1, \dots, i_m}|^{r_m} \right)^{\frac{1}{r_m}} \cdots \right)^{\frac{1}{r_3} \times r_2} \right)^{\frac{1}{r_2} \times r_1} \right)^{\frac{1}{r_1}} \\ & \leq C_{m,p} \cdot n^{\left(\sum_{j=1}^m \frac{1}{r_j}\right) - \frac{m}{2}} \left(\int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^n r_{i_1}(t_1) \cdots r_{i_m}(t_m) a_{i_1, \dots, i_m} \right|^p dt_1 \cdots dt_m \right)^{1/p} \end{aligned}$$

and that the exponent $\left(\sum_{j=1}^m \frac{1}{r_j}\right) - \frac{m}{2}$ is sharp.

4. OPTIMAL CONSTANTS FOR VARIANTS OF THE KHINCHIN INEQUALITY

We begin this section by providing the optimal constants satisfying (1.2) when $p = 1$ and $r \geq 2$:

Proposition 1. Let m, n be positive integers and $(a_{i_1, \dots, i_m})_{i_1, \dots, i_m=1}^n$ be a sequence of real scalars. If $r \geq 2$, then

$$(4.1) \quad \left(\sum_{i_1, \dots, i_m=1}^n |a_{i_1, \dots, i_m}|^r \right)^{\frac{1}{r}} \leq 2^{\frac{m}{r}} \int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^n r_{i_1}(t_1) \cdots r_{i_m}(t_m) a_{i_1, \dots, i_m} \right| dt_1 \cdots dt_m$$

and the estimate $2^{\frac{m}{r}}$ is optimal.

Let us denote by C_r the optimal constant satisfying

$$(4.2) \quad \left(\sum_{i_1, \dots, i_m=1}^n |a_{i_1, \dots, i_m}|^r \right)^{\frac{1}{r}} \leq C_r \int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^n r_{i_1}(t_1) \cdots r_{i_m}(t_m) a_{i_1, \dots, i_m} \right| dt_1 \cdots dt_m$$

for all sequence of scalars $(a_{i_1, \dots, i_m})_{i_1, \dots, i_m=1}^n$, for all n . Let $\theta = \frac{2}{r}$; by the Hölder inequality, (1.2) and Lemma 1 we conclude that

$$\begin{aligned} \left(\sum_{i_1, \dots, i_m=1}^n |a_{i_1, \dots, i_m}|^r \right)^{\frac{1}{r}} & \leq \left(\sum_{i_1, \dots, i_m=1}^n |a_{i_1, \dots, i_m}|^2 \right)^{\frac{\theta}{2}} \cdot \left(\max_{\substack{i_k=1, \dots, n \\ k=1, \dots, m}} |a_{i_1, \dots, i_m}| \right)^{1-\theta} \\ & \leq 2^{\frac{m\theta}{2}} \int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^n r_{i_1}(t_1) \cdots r_{i_m}(t_m) a_{i_1, \dots, i_m} \right| dt_1 \cdots dt_m \\ & = 2^{\frac{m}{r}} \int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^n r_{i_1}(t_1) \cdots r_{i_m}(t_m) a_{i_1, \dots, i_m} \right| dt_1 \cdots dt_m, \end{aligned}$$

Now let us prove that the constant $2^{\frac{m}{r}}$ is sharp. Let R_{m+1} be the $m+1$ -linear form defined in the Section 2. Using that $(c_0)^* = \ell_1$, we have

$$\begin{aligned}
& \sum_{i_{m+1}=1}^{2^m} \left(\sum_{i_1, \dots, i_m=1}^{2^m} |R_{m+1}(e_{i_1}, e_{i_2}, \dots, e_{i_{m+1}})|^r \right)^{\frac{1}{r}} \\
& \leq \sum_{i_{m+1}=1}^{2^m} C_r \int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^{2^m} r_{i_1}(t_1) \cdots r_{i_m}(t_m) R_{m+1}(e_{i_1}, e_{i_2}, \dots, e_{i_{m+1}}) \right| dt_1 \cdots dt_m \\
& = C_r \int_{[0,1]^m} \sum_{i_{m+1}=1}^{2^m} \left| R_{m+1} \left(\sum_{i_1=1}^{2^m} r_{i_1}(t_1) e_{i_1}, \dots, \sum_{i_m=1}^{2^m} r_{i_m}(t_m) e_{i_m}, e_{i_{m+1}} \right) \right| dt_1 \cdots dt_m \\
& \leq C_r \sup_{t_1, \dots, t_m \in [0,1]} \sum_{i_{m+1}=1}^{2^m} \left| R_{m+1} \left(\sum_{i_1=1}^{2^m} r_{i_1}(t_1) e_{i_1}, \dots, \sum_{i_m=1}^{2^m} r_{i_m}(t_m) e_{i_m}, e_{i_{m+1}} \right) \right| \\
& \leq 2^m C_r.
\end{aligned}$$

On the other hand, since R_{m+1} has exactly 2^m monomials involving the coordinates of the last variable and since R_{m+1} has a total of 2^{2m} monomials, we conclude that

$$\sum_{i_{m+1}=1}^{2^m} \left(\sum_{i_1, \dots, i_m=1}^{2^m} |R_{m+1}(e_{i_1}, e_{i_2}, \dots, e_{i_{m+1}})|^r \right)^{\frac{1}{r}} = 2^m \cdot (2^m)^{\frac{1}{r}}.$$

Thus,

$$2^m \cdot 2^{\frac{m}{r}} \leq 2^m C_r$$

and we obtain

$$C_r \geq 2^{\frac{m}{r}},$$

completing the proof.

Remark 2. *It sounds reasonable that there exists a more direct proof of Proposition 1. However, the fact that in general $\left(\prod_{j=1}^m r_{i_j} : [0,1]^m \rightarrow \mathbb{R} \right)_{i_1, \dots, i_m=1}^\infty$ is not independent may be an additional difficulty.*

5. BLOW UP RATE OF KAHANE TYPE INEQUALITIES

Let $2 \leq q < \infty$ and $s > 0$. A Banach space Y has cotype q (see [5, 12]) if there is a constant $C > 0$ such that, no matter how we select finitely many vectors $y_1, \dots, y_n \in Y$,

$$(5.1) \quad \left(\sum_{k=1}^n \|y_k\|^q \right)^{\frac{1}{q}} \leq C \left(\int_{[0,1]} \left\| \sum_{k=1}^n r_k(t) y_k \right\|^s dt \right)^{\frac{1}{s}}.$$

The smallest of all these constants is denoted by $C_q(Y)$ when $s = 2$ and $c_q(Y)$ when $s = q$. The Kahane inequality (below) shows that the choice of s is not relevant (modulo the constant involved):

Theorem 3 (Kahane Inequality). *If $0 < p, q < \infty$, then there is a constant $K_{p,q} > 0$ for which*

$$\left(\int_{[0,1]} \left\| \sum_{k=1}^n r_k(t) y_k \right\|^q dt \right)^{\frac{1}{q}} \leq K_{p,q} \left(\int_{[0,1]} \left\| \sum_{k=1}^n r_k(t) y_k \right\|^p dt \right)^{\frac{1}{p}}$$

holds, regardless of the choice of a Banach space Y and of finitely many vectors $y_1, \dots, y_n \in Y$.

From now on $K_{p,q}$ denotes the optimal constant of the Kahane inequality. As it happens for the Khinchin inequality, we have a Kahane inequality for multiple indexes (see, for instance, [1]):

Theorem 4 (Multiple Kahane Inequality). *If $0 < p, q < \infty$, then*

$$\begin{aligned} & \left(\int_{[0,1]^m} \left\| \sum_{i_1, \dots, i_m=1}^n y_{i_1, \dots, i_m} r_{i_1}(t_1) \cdots r_{i_m}(t_m) \right\|^q dt_1 \cdots dt_m \right)^{\frac{1}{q}} \\ & \leq K_{p,q}^m \left(\int_{[0,1]^m} \left\| \sum_{i_1, \dots, i_m=1}^n y_{i_1, \dots, i_m} r_{i_1}(t_1) \cdots r_{i_m}(t_m) \right\|^p dt_1 \cdots dt_m \right)^{\frac{1}{p}}, \end{aligned}$$

for all Banach spaces Y and all y_{i_1, \dots, i_m} in Y .

The following result shows how cotype q spaces behave with sums in multiple indexes (see, for instance, [12, Lemma 3.9]):

Theorem 5 (Multiple cotype inequality). *Let Y be a cotype q space. If $(y_{i_1 \dots i_m})_{i_1, \dots, i_m=1}^n$ is a matrix in Y , then*

$$\left(\sum_{i_1, \dots, i_m=1}^n \|y_{i_1 \dots i_m}\|^q \right)^{1/q} \leq c_q(Y)^m \left(\int_{[0,1]^m} \left\| \sum_{i_1, \dots, i_m=1}^n y_{i_1 \dots i_m} r_{i_1}(t_1) \cdots r_{i_m}(t_m) \right\|^q dt_1 \cdots dt_m \right)^{1/q}.$$

By the multiple Kahane inequality it is plain that from the above inequality we have

$$\begin{aligned} (5.2) \quad & \left(\sum_{i_1, \dots, i_m=1}^n \|y_{i_1 \dots i_m}\|^q \right)^{1/q} \\ & \leq c_q(Y)^m K_{s,q}^m \left(\int_{[0,1]^m} \left\| \sum_{i_1, \dots, i_m=1}^n y_{i_1 \dots i_m} r_{i_1}(t_1) \cdots r_{i_m}(t_m) \right\|^s dt_1 \cdots dt_m \right)^{\frac{1}{s}} \end{aligned}$$

for all $s > 0$. Our next result shows how is the exact blow up rate of the constant arising when we consider cotype 2 spaces replacing the ℓ_2 norm by a ℓ_r norm, $r < 2$, in the left hand side of the above inequality.

Theorem 6. *Let $Y \neq \{0\}$ be a cotype 2 space and $p > 0$. If $0 < r \leq 2$ and $(y_{i_1 \dots i_m})_{i_1, \dots, i_m=1}^n$ is a matrix in Y , then there is a constant $c_{m,p} > 0$ such that*

$$\begin{aligned} & \left(\sum_{i_1, \dots, i_m=1}^n \|y_{i_1 \dots i_m}\|^r \right)^{1/r} \\ & \leq c_{m,p} n^{m(\frac{1}{r} - \frac{1}{2})} \left(\int_{[0,1]^m} \left\| \sum_{i_1, \dots, i_m=1}^n y_{i_1 \dots i_m} r_{i_1}(t_1) \cdots r_{i_m}(t_m) \right\|^p dt_1 \cdots dt_m \right)^{\frac{1}{p}} \end{aligned}$$

and the exponent $m(\frac{1}{r} - \frac{1}{2})$ is optimal.

Proof. As in the proof of Theorem 1, we have

$$\begin{aligned} (5.3) \quad & \left(\sum_{i_1, \dots, i_m=1}^n \|y_{i_1, \dots, i_m}\|^r \right)^{\frac{1}{r}} \\ & \leq c_2(Y)^m K_{1,2}^m n^{m(\frac{1}{r} - \frac{1}{2})} \cdot \int_{[0,1]^m} \left\| \sum_{i_1, \dots, i_m=1}^n r_{i_1}(t_1) \cdots r_{i_m}(t_m) y_{i_1, \dots, i_m} \right\| dt_1 \cdots dt_m. \end{aligned}$$

To prove the optimality of the above exponent $m\left(\frac{1}{r} - \frac{1}{2}\right)$, let us suppose that

$$\left(\sum_{i_1, \dots, i_m=1}^n \|y_{i_1, \dots, i_m}\|^r \right)^{\frac{1}{r}} \leq c_m n^t \cdot \int_{[0,1]^m} \left\| \sum_{i_1, \dots, i_m=1}^n r_{i_1}(t_1) \cdots r_{i_m}(t_m) y_{i_1, \dots, i_m} \right\| dt_1 \cdots dt_m$$

for a certain $c_m > 0$. Consider the $m+1$ -linear form $T_{m+1,n}$ given by the Kahane–Salem–Zygmund inequality and define

$$S_{m+1,n}(x_1, \dots, x_{m+1}) = T_{m+1,n}(x_1, \dots, x_{m+1})y,$$

for a certain fixed $y \in Y$ with $\|y\| = 1$. Then

$$\begin{aligned} & \sum_{i_{m+1}=1}^n \left(\sum_{i_1, \dots, i_m=1}^n \|S_{m+1,n}(e_{i_1}, \dots, e_{i_{m+1}})\|^r \right)^{\frac{1}{r}} \\ & \leq \sum_{i_{m+1}=1}^n c_m n^t \int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^n r_{i_1}(t_1) \cdots r_{i_m}(t_m) T_{m+1,n}(e_{i_1}, e_{i_2}, \dots, e_{i_{m+1}}) \right| dt_1 \cdots dt_m \\ & \leq c_m n^t K_{m+1} n^{\frac{m+2}{2}}. \end{aligned}$$

Proceeding again as in the proof of Theorem 1 we conclude that

$$t \geq m \left(\frac{1}{r} - \frac{1}{2} \right).$$

By Theorem 4 we know that the same optimal estimate holds when replacing the L_1 -norm in (5.3) by any L_p -norm. \square

Remark 3. A result similar to the one stated in Remark 1 applies for this case of cotype 2 spaces.

When Y is a Hilbert space we can prove a result similar to Proposition 1:

Theorem 7. Let m, n be positive integers and $(y_{i_1, \dots, i_m})_{i_1, \dots, i_m=1}^n$ be a sequence in a Hilbert space Y . If $r \geq 2$, then

$$\left(\sum_{i_1, \dots, i_m=1}^n \|y_{i_1, \dots, i_m}\|^r \right)^{\frac{1}{r}} \leq 2^{\frac{m}{r}} \int_{[0,1]^m} \left\| \sum_{i_1, \dots, i_m=1}^n r_{i_1}(t_1) \cdots r_{i_m}(t_m) y_{i_1, \dots, i_m} \right\| dt_1 \cdots dt_m$$

and the constant $2^{\frac{m}{r}}$ is optimal.

Proof. The proof is similar to the proof of Proposition 1. For $r \geq 2$ let us denote by C_r the optimal constant satisfying

$$\left(\sum_{i_1, \dots, i_m=1}^n \|y_{i_1, \dots, i_m}\|^r \right)^{\frac{1}{r}} \leq C_r \int_{[0,1]^m} \left\| \sum_{i_1, \dots, i_m=1}^n r_{i_1}(t_1) \cdots r_{i_m}(t_m) y_{i_1, \dots, i_m} \right\| dt_1 \cdots dt_m$$

for all sequence of scalars $(y_{i_1, \dots, i_m})_{i_1, \dots, i_m=1}^n$, for all n . Let $\theta = \frac{2}{r}$; since $K_{1,2} = \sqrt{2}$ and $c_2(Y) = 1$ (see [9] and [5, Corollary 11.8]), by the Hölder inequality, (5.2) and Lemma 1 we conclude that

$$\begin{aligned} \left(\sum_{i_1, \dots, i_m=1}^n \|y_{i_1, \dots, i_m}\|^r \right)^{\frac{1}{r}} &\leq \left(\sum_{i_1, \dots, i_m=1}^n \|y_{i_1, \dots, i_m}\|^2 \right)^{\frac{\theta}{2}} \cdot \left(\max_{\substack{i_k=1, \dots, n \\ k=1, \dots, m}} \|y_{i_1, \dots, i_m}\| \right)^{1-\theta} \\ &\leq (c_2(Y)^m K_{1,2}^m)^\theta \int_{[0,1]^m} \left\| \sum_{i_1, \dots, i_m=1}^n r_{i_1}(t_1) \cdots r_{i_m}(t_m) y_{i_1, \dots, i_m} \right\| dt_1 \cdots dt_m \\ &= 2^{\frac{m}{r}} \int_{[0,1]^m} \left\| \sum_{i_1, \dots, i_m=1}^n r_{i_1}(t_1) \cdots r_{i_m}(t_m) y_{i_1, \dots, i_m} \right\| dt_1 \cdots dt_m, \end{aligned}$$

Now let us prove that the constant $2^{\frac{m}{r}}$ is sharp. Let S_{m+1} be the $m+1$ -linear form R_{m+1} defined in Section 2, multiplied by a fixed unit vector $y \in Y$. Using that $(c_0)^* = \ell_1$, we have

$$\begin{aligned} &\sum_{i_{m+1}=1}^{2^m} \left(\sum_{i_1, \dots, i_m=1}^{2^m} \|S_{m+1}(e_{i_1}, e_{i_2}, \dots, e_{i_{m+1}})\|^r \right)^{\frac{1}{r}} \\ &\leq \sum_{i_{m+1}=1}^{2^m} C_r \int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^{2^m} r_{i_1}(t_1) \cdots r_{i_m}(t_m) R_{m+1}(e_{i_1}, e_{i_2}, \dots, e_{i_{m+1}}) \right| dt_1 \cdots dt_m \\ &= C_r \int_{[0,1]^m} \sum_{i_{m+1}=1}^{2^m} \left| R_{m+1} \left(\sum_{i_1=1}^{2^m} r_{i_1}(t_1) e_{i_1}, \dots, \sum_{i_m=1}^{2^m} r_{i_m}(t_m) e_{i_m}, e_{i_{m+1}} \right) \right| dt_1 \cdots dt_m \\ &\leq C_r \sup_{t_1, \dots, t_m \in [0,1]} \sum_{i_{m+1}=1}^{2^m} \left| R_{m+1} \left(\sum_{i_1=1}^{2^m} r_{i_1}(t_1) e_{i_1}, \dots, \sum_{i_m=1}^{2^m} r_{i_m}(t_m) e_{i_m}, e_{i_{m+1}} \right) \right| \\ &\leq 2^m C_r. \end{aligned}$$

On the other hand, since R_{m+1} has exactly 2^m monomials involving the coordinates of the last variable and since R_{m+1} has a total of 2^{2m} monomials, we conclude that

$$\sum_{i_{m+1}=1}^{2^m} \left(\sum_{i_1, \dots, i_m=1}^{2^m} \|R_{m+1}(e_{i_1}, e_{i_2}, \dots, e_{i_{m+1}}) y\|^r \right)^{\frac{1}{r}} = 2^m \cdot (2^m)^{\frac{1}{r}}.$$

and the proof is concluded as in Proposition 1. \square

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